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Spatial Geometry Learning in Higher Education Through Flipped Classroom with Metacognitive Scaffolding: A Mixed-Methods Study

Awi* Universitas Negeri Makassar, INDONESIA

Sutamrin Universitas Negeri Makassar, INDONESIA

Hisyam Ihsan Universitas Negeri Makassar, INDONESIA

Norma Nasir Universitas Negeri Makassar, INDONESIA

Muhammad Ammar Naufal Universitas Negeri Makassar, INDONESIA

Muhammad Ikram Universitas Negeri Makassar, INDONESIA

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Abstract

Mathematics education students continue to struggle with the complex notion of spatial geometry. Therefore, learning approaches should incorporate strong conceptual understanding. This study examined the effectiveness of flipped classroom-based geometry learning combined with metacognitive scaffolding in developing students' spatial ability. It also identified students' spatial knowledge needs, types of metacognitive questions posed by the teacher, and forms of scaffolding used. A mixed-methods design was employed, consisting of an exploratory qualitative phase followed by a quasi-experimental phase. The qualitative phase explored key spatial knowledge demands and scaffolding patterns through questionnaires and interviews, while the quasi-experimental phase compared three instructional conditions: flipped classroom with metacognitive scaffolding, flipped classroom without scaffolding, and conventional instruction. Results indicated that students needed four types of spatial geometry knowledge: 3D coordinate representation, geometric transformation, spatial visualization, and angle–distance relationships. Teachers' metacognitive questions mainly emphasized awareness of understanding, while conceptual scaffolding was more dominant than metacognitive scaffolding. Quantitatively, students in the flipped classroom with metacognitive scaffolding showed greater improvements in spatial ability. These findings suggest integrating self-directed learning and metacognitive support in geometry learning that requires advanced visualization and independent thinking.

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INTRODUCTION

Mathematics education students face an actual challenge in mastering the aspects of spatial geometry, most obviously when it concerns 3D coordinates, visualization of shapes, mapping spatial transformations, and the interrelations of angles, distances, and other spatial elements (Borji & Martínez-Planell, 2023; David et al., 2018; Rellensmann et al., 2017). The reason for these difficulties is an insufficient spatial representation skill, i.e., an ability to mentally represent pictures of three-dimensional forms and correlate numerical information with visual meanings (Cory & Garofalo, 2011; Haciomeroglu et al., 2010; Krawec, 2014). Yet it is a tendency they rely rather on algebraic techniques and do not sufficiently grasp the spatial importance of the developed geometric construction itself (Haciomeroglu & LaVenia, 2017; Pitta-Pantazi et al., 2020). Such a

* **Corresponding Author**
Awi,, Universitas Negeri Makassar, INDONESIA ✉ awi.dassa@unm.ac.id

tendency is further worsened by poor opportunities for exploration of spatial objects using manipulatives and vision technology (Haciomeroglu, 2015).

These visualization gaps, in turn, impact students' ability to solve spatial geometry problems, especially contextual geometry problems, which demand the integration of symbolic, graphic, and spatial representations (Minh & Lagrange, 2016; Walkington et al., 2024). Existing literature has identified the importance of spatial visualization skills, which are paramount in dealing with complex geometry problems, particularly in situations where students are asked to relate abstract mathematical concepts and concrete visualization in space (Bragelman et al., 2024; Gasteiger et al., 2020; Saleh et al., 2018). In other situations, the inability to create meaningful mental images of the given geometry prevents students from reverting to procedural or arithmetic strategies, which results in mechanical thinking and the inability to grasp key concepts (Chang et al., 2016; Cuevas-Vallejo et al., 2023; Rach & Ufer, 2020; Robinson & LeFevre, 2012; Scheibling-Sève et al., 2020). Thus, geometry should be taught in a manner that promotes good visualization and spatial visualization.

Another recent teaching strategy that is gaining popularity and seems to be effective, especially when time and interaction are critical, mainly in relation to geometry classes, is the "flipped classroom" strategy. As described, it's quite simple: students will be asked to complete the main aspects of a lesson before the actual classroom-based interactions. As might be expected, some research suggests that "flipped" teaching is not automatically effective for developing "deep" spatial knowledge and skills, mainly when it promotes the simple delivery of knowledge without metacognitive reflection (Cevikbas & Kaiser, 2020; Cronhjort et al., 2018; Fredriksen, 2021; Helgevold & Moen, 2015). In spatial geometry, where visualization is critical, "flipped" approaches need to be incorporated with effective classroom approaches that allow students to develop awareness of their own cognitive processes (Voigt et al., 2020).

Metacognitive scaffolding is another promising research avenue for addressing the above-mentioned information gap. In fact, it has been suggested that it is helpful to teach students, in a quite deliberate manner, to monitor, evaluate, and reflect on their own cognitive processes while dealing with problems (Cevikbas & Kaiser, 2020; Kosko, 2020), at least in geometry classes. Here, more specific questions such as "Why have you adopted this strategy?" or "Are there any other ways of tackling your problem?" can prompt students to reflect more attentively, make adjustments in their comprehension, and eventually link what they learn with possible real-world applications. Observing classrooms, however, it is clear that several instructors still make use of "Do you understand?" type questions, which mostly revolve around whether students feel they can grasp something or not (Hein & Prediger, 2024; Schoonen et al., 2011).

Such teachers rely more on conceptual and procedural scaffolding but pay less attention to metacognitive scaffolding, enabling students to become independent thinkers. Procedural and conceptual scaffolding facilitate student problem-solving and comprehending geometric concepts and rules; they do not promote self-assessing students and their learning (Nagle et al., 2019). The metacognitive scaffolding technique, as mentioned above, helps increase students' critical thinking and options in choosing strategies in solving geometry problems (Finesilver, 2022; Rellensmann et al., 2017; Soneira et al., 2018).

These findings highlight the importance of mixing the flipped classroom model with metacognitive scaffolding to enhance students' spatial thinking in geometry. The flipped classroom model allows students to explore the content before class, whereas metacognitive scaffolding enables reflection, monitoring, and evaluation to guide students with their strategies in solving problems during class,(Kosko, 2020; Leron & Paz, 2014; Speer & Wagner, 2009). This can lead to even better meaning-making in terms of spatial thinking and better performance in complex spatial geometry problems.

Despite the increasing interest in flipped classroom approaches and scaffolding strategies in math education, a number of research gaps remain. Firstly, few controlled or quasi-experimental studies have explored how flipping the classroom, coupled with metacognitive scaffolding, impacts learning in spatial geometry, especially at higher education. Secondly, only a handful of studies have provided a detailed analysis of the types of metacognitive questions and scaffolding practices that teachers actually employ in the course of geometry lessons. Thirdly, there has been limited empirical work conducted to systematically identify students' needs concerning their spatial knowledge to inform instructional design. These methodological gaps collectively point to the need for integrative research that links learning outcomes with instructional processes.

What is novel about this study is its deliberate combination of flipped classroom techniques and metacognitive scaffolding within the context of college-level spatial geometry education. Contrasted with the prior work that tends to study flipped classrooms or scaffolding in isolation, or that centers on engagement and achievement, this research brings the learning gains in instructional mechanisms of spatial geometry to the fore. It does so by examining, in concert, students' spatial knowledge demands, the metacognitive scaffolding practices instructors actually implement, and the consequent learning gains across different conditions.

In consideration of this, therefore, this research must aim to explore the extent to which a flipped classroom, particularly with enhanced metacognitive scaffolding, can promote spatial skills among university students, as well as identify what students need to know in spatial concepts, and what instructors aim at as far as metacognitive scaffolding is concerned within geometry classes. The research design of this study, which is a mixed design with qualitative as well as a quasi-experiment, provides a comprehensive understanding of learning processes as well as learning outcomes.

Further developing our understanding of geometry education that incorporates flipped learning and metacognition, two basic aspects will be highlighted; (1) What are the students' needs for understanding geometric concepts? What are the nature and form of metacognitive prompts and scaffolding that teachers need to design for developing students' geometric spatial skills?; (2) How effective is the combination of Flipped Classroom and metacognitive scaffolding for the teaching of geometry?

METHOD

This investigation aimed to monitor the development of students' spatial geometry abilities under varying conditions, while also exploring the needs of students and how teachers facilitate them through a qualitative investigation. Furthermore, as it is always advisable to remain in sync with the investigation queries, design, and analysis process, a mixed-inquiry method, in this case, proved to be useful. Most importantly, on one hand, qualitative inquiry aided in making explicit discernments regarding spatial geometry abilities that students lacked, as well as queries in metacognition and scaffolding guided by teachers. On the other hand, a quasi-experimental design with pre-tests and post-tests involving control groups evaluated the development of students' spatial geometry abilities under respective conditions to determine whether a flipped classroom with metacognitive scaffolding aided in improvements in geometry abilities among students. We implemented three different interventions in three classes; class A studied geometry using the flipped classroom and the metacognitive scaffolding approach, class B utilized a conventional learning method, and class C utilized the flipped classroom without metacognitive scaffolding. The research design is shown in Table 1.

Table 1. Research Design

Group	Pre-test	Intervention	Post-Test
Class A	O_1	The flipped classroom with metacognitive scaffolding	O_2
Class B	O_1	Conventional method	O_2
Class C	O_1	The flipped classroom without metacognitive scaffolding	O_2

Participant

This study involved three groups of students with diverse backgrounds (n = 92) (age 18 – 20 years) (participants' demography is shown in Table 2). We invited three geometry lecturers to teach each class. Before performing the experiment, we made sure that the lecturers were briefed about the different approaches they were going to implement in their classrooms. We tried to convince the lecturers to adjust the teaching approach to the needs of their students. We scheduled the lessons for two hours per week and provided the students with the appropriate learning materials. One intriguing component of the learning design developed in this study was that the students could choose among three simultaneous modes of learning process, namely: (1) traditional face-to-face sessions; (2) asynchronous online learning featuring videos with content, pedagogy, and demonstrations via Geogebra; and (3) a combined version of the former modes (hybrid). All students involved in this research were required to complete the same assignments and provide comments on the learning process during the same time interval.

Table 2. Participants' Demography

Aspect	Number	Percentage
Gender		
Men	35	38%
Women	57	62%
Group		
Class A	29	31%
Class B	33	36%
Class C	30	33%
Participants' place of origin		
Urban area	21	23%
Suburban area	54	58%
Rural area	27	29%
GPA		
GPA$\geq$ 3.75	18	19%
3.5 $\leq$ GPA$<$ 3.75	34	37%
GPA $<$ 3.5	40	44%

Lecturers who participated in collaborative professional learning with the researchers (the authors of this paper) focused on helping the students develop spatial ability in geometry learning. They were given handouts and materials containing a series of tasks, including assessments in the form of pre-tests and post-tests. During the initial planning, the lecturers assigned geometry tasks related to angles between spaces and distances between spatial elements to the students. The tasks were developed by the researchers by prioritizing theoretical perspectives and empirical findings that have been previously described in existing literature. The tasks given for each session are presented in Tables 3 and 4.

Table 3. Spatial Task Design

Task	Spatial Topic	Problem Description
Task 1	Angles Between Spatial Elements	A cuboid (ABCD.EFGH) has the following dimensions: AB = 6 cm, BC = 8 cm, BF = 4 cm. Let *a* be the angle between AH and BD. Determine cos 2a.
Task 2	Distances Between Spatial Elements	A cube (ABCD.EFGH) has an edge length of 5 cm. M is the intersection of AF and BE. N is the midpoint of EH. Determine the distance between BH and MN.

Table 4. Spatial Ability Components Measured

Task	Indicator Code	Spatial Ability Component Measured
T1	SA1	Understanding how to determine space diagonals in a rectangular prism
T1	SA2	Applying the Pythagorean theorem to calculate diagonal lengths
T1	SA3	Using vector dot product to determine the angle between two lines in space
T1	SA4	Applying trigonometric identity to determine cos 2a
T1	SA5	Visualizing the relationship between skew diagonals in 3D space
T2	SA6	Identifying coordinates of cube vertices
T2	SA7	Determining the intersection of lines in three-dimensional space

Task	Indicator Code	Spatial Ability Component Measured
T2	SA8	Applying the midpoint formula in 3D geometry
T2	SA9	Visualizing spatial relationships between elements in a cube
T2	SA10	Determining distances between spatial elements

Instrument

This study employed two instruments in data collection. First, an open-ended questionnaire was utilized to identify the knowledge needed by students in understanding concepts in geometry. The questionnaire was distributed to and completed by the students. We also conducted open-ended interviews with the lecturers to map metacognitive questions and forms of scaffolding used to help the students solve geometry problems. Second, a learning outcome test was performed to measure the effectiveness of the flipped classroom combined with a metacognitive scaffolding approach on students' spatial ability.

The learning outcome test was developed based on established theoretical frameworks of spatial ability, including three-dimensional visualization, spatial relations, and geometric transformations. Content validity was examined through expert judgment involving three mathematics education experts, who evaluated the relevance, clarity, and representativeness of test items. Revisions were made based on their feedback before the instrument was administered. Sample indicators included students' ability to determine angles between spatial elements, visualize non-coplanar diagonals, and calculate distances between lines and points in three-dimensional space.

Procedures

We conducted the study in two stages: the qualitative exploratory stage and the quantitative experimental stage. At the first stage, a qualitative approach was used to identify students' needs in understanding spatial geometry concepts, as well as the forms of metacognitive questions and scaffolding used by the instructor. The process of collecting data involved using a student questionnaire, lecturer interviews, and observations. The collected data were analyzed to identify important knowledge types for geometry instruction, problems faced by students, and teaching guidance provided. In the second stage, quasi-experimental research, pre- and post-test, and control groups were applied. There were 92 participants, divided into three groups. Class A practiced Flipped Classroom learning and metacognitive guidance, Class B practiced traditional teaching methods, and Class C practiced Flipped Classroom without guidance. The research process lasted for several weeks, depending on the teaching plans for these classes. Learning materials, teaching instructions, and learning tasks were standardized. Students from all classes had access to three different media for learning, i.e., face-to-face, asynchronous online (through interactive video formats), and a combination of both. All students completed learning tasks, recorded their learning processes, and ensured that every student had access to strategy implementation skills through three professional lecturers, who were expert practitioners in using strategies for their respective classes.

Analysis

What did we do? We combined two different approaches: qualitative and quantitative analysis. When it came to qualitative analysis, we focused on patterns in students' understanding of spatial geometry, types of metacognitive questions used by the lecturer, and types of scaffolding used to support students. We used coding to analyze open-ended survey answers, interview transcriptions, and notes to identify major themes related to students, their needs, their struggles, and effectiveness in supporting students with understanding spatial concepts. On the quantitative analysis side, we performed analysis using descriptive and inferential statistics. We performed a comparison between students' scores, using the pre-test and post-test. We also calculated the Normalized Gain (N-Gain) to measure the relative improvement in learning achievement. To ensure rigors in this quantitative analysis, we performed tests on normality using the Shapiro-Wilk and homogeneity of variance using Levene's test. We carried out a difference test between groups using an independent t-test. We also carried out a post-hoc using the Tukey HSD to check the differences.

According to the analysis results, the group using a flipped classroom with metacognitive scaffolding (Class A) demonstrated the highest increase in post-test scores and N-Gain, significantly different from the other groups. This also showed that by utilizing metacognitive scaffolding in flipped instruction, students could understand spatial geometry concepts much better. To complement statistical significance testing, effect size was calculated using Cohen's d to determine the magnitude of differences between instructional conditions. The inclusion of effect size provides a more comprehensive interpretation of instructional effectiveness beyond p-values alone.

RESULTS AND DISCUSSION

Q1: What do students need to understand concepts in geometry? What kind of metacognitive questions do teachers need to develop to instill geometric spatial skills in students, and what form of scaffolding should they take??

The findings of the questionnaire analysis and observations made regarding students' completion of tasks indicated that students required four different kinds of knowledge to understand the geometric concepts presented in Figure 1.

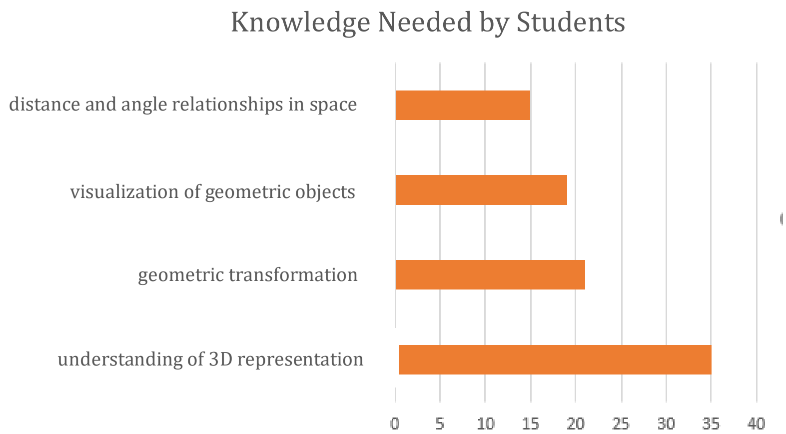

Figure 1. Knowledge needed by Students in Geometry Learning based on the Questionnaire's Response Analysis

The results of the questionnaire analysis showed that 39% of students responded favorably to the statement "an understanding of 3D coordinates is required to comprehend how a point is represented in three-dimensional space (x, y, z)". Most students concurred that knowing 3D coordinates is necessary in a number of situations, including figuring out where a point is on a plane or how it is projected onto a line and a plane. Students reported having trouble converting information from algebraic form to visual representation and visualizing the relationship between points in a 3D coordinate system. To minimize these problems, teachers can use software or concrete manipulatives to help students understand how points and coordinates interact in space. The following statement from a student supports this finding:

" I can understand the coordinates of a point, but it's hard to imagine how a diagonal line in a block fits in three-dimensional space. "

Twenty-one percent (21%) of respondents agreed that knowledge about geometric transformations can be used to overcome the difficulty in changing coordinates after moving a point or shape in three-dimensional space. In this case, students need to understand how an object changes its position in space due to translation, rotation, or reflection. To minimize this difficulty, students must be given dynamic geometry-based exercises using software such as Geo*Gebra, SketchUp,* or *Wolfram Alpha* that facilitate understanding of how transformations occur interactively. In addition, students need to be given tutorials on how to manipulate virtual objects in 3D to see their coordinates change due to transformation. A student's statement supporting this finding is presented below.

" I am still confused when the point is moved 3 units to the right. I have no idea how to change its coordinates in a 3D system."

Nineteen percent (19%) of respondents thought that they had difficulty in imagining spatial geometric shapes, such as cubes, blocks, inclined planes, and spatial diagonals. In other words, they struggled to comprehend the perspective of an object, particularly when it came to tasks that involve angles between lines, distances from points to lines, or points to planes. In such circumstances, students with less visualization power face difficulties in visualizing the data provided, either in oral or written form, and making it easier to comprehend in their minds. In response to these issues, learners may be provided with more activities centered on the basics of actual image representations or with augmented reality to enhance their visualization power. Students should be provided with opportunities to practice drawing three-dimensional shapes either manually or through graphical representations. The following represents the perspective of a respondent about the importance of visualizing geometric objects.

" I have difficulty with the relationship between angles on lines, the distance between points and lines, or points to planes."

Around 15% of the population highlighted the importance of constructing knowledge on the relationship between space and distance/angle, as this is fundamental in understanding how certain elements of geometry fit together. The difficulty experienced by students in this section lies in how they relate mathematical formulas to their spatial understanding. To overcome this challenge, students can be provided with a visual model that can show how angles are formed between two lines or two planes in space, either through manual diagrams or digital animations. The following is an excerpt from a student interview related to this finding.

"I can calculate the length of the interior diagonal of a cuboid, but I have no idea how to determine the angle between two diagonals."

In general, of the four types of geometric knowledge shown in Figure 1, understanding 3D coordinates is the most crucial knowledge that students need for developing spatial ability and is the key to mastering all spatial geometry concepts. Calculating distances and angles, as well as visualizing forms and object changes in space, remain the primary challenges that students face in geometry learning. Therefore, integrating technology-based approaches into geometry classes, such as interactive geometry software and 3D simulations, can help strengthen students' spatial ability through interactive and problem-solving-based learning strategies. After determining the types of geometric knowledge needed by students in geometry learning, we identified the forms of metacognitive questions frequently asked by the lecturer during the lesson. Figure 2 illustrates the distribution of metacognitive question types posed during the geometry instruction.

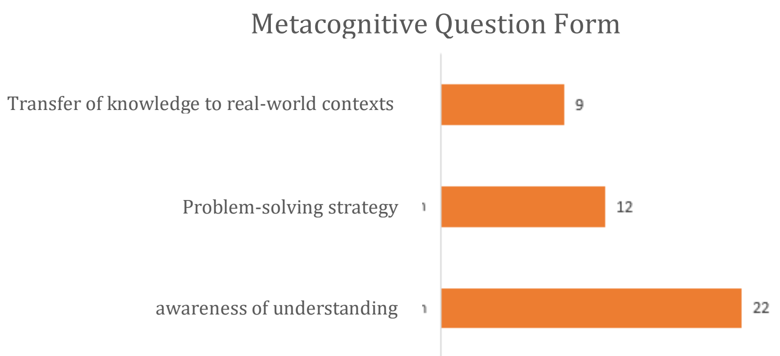

Figure 2. Metacognitive Questions asked by the Lecturer during the Lesson

According to this study's findings, lecturers frequently posed questions that can help students realize how far they understand the concepts being taught. In this case, 51% of the activities carried out by the lecturers are related to metacognitive questions that promote students' awareness of whether they have understood a concept. This indicates that the lecturers have been trying to stimulate students to identify parts of the learning that students have or have not understood. With this stimulation, students can recognize whether they need to overcome difficulties in visualizing spatial geometry objects. Therefore, the lecturers must encourage more individual reflection through group discussions and/or using visual aids to minimize students' difficulties in understanding spatial representations. The following is an example of metacognitive questions asked by the lecturers to awaken students' awareness of their geometry understanding.

Lecturer	*How did you find out that the two calculated diagonals lie in three-dimensional space?* ***(#Task1)***
Student 1	*"I knew that the diagonal BD lies on the base plane of the cuboid, and AH is the diagonal of one of the vertical sides of the cuboid. But I was confused about how to determine the angle between the two, because they do not lie in the same plane."*
Student 2	*" I applied the Pythagorean theorem to calculate the length of the diagonals BD and AH, but I did not understand how to determine the angle between them. I am aware that an angle is formed when two lines meet, but how can I figure that out?"*

Furthermore, 28% of metacognitive questions posed by the lecturers were aimed at assisting students in choosing, comparing, and evaluating their problem-solving strategy. The observation results showed that students struggled to select the most efficient strategy for calculating distance or angle in three-dimensional space. To overcome these difficulties, lecturers can ask students to compare several problem-solving methods in groups and use interactive technology or applications to explore various visual strategies. An example of metacognitive questions that were aimed at helping students select, compare, and evaluate their problem-solving strategy is presented below.

Lecturer	*How did you make sure that point M is really the intersection of AF and BE??* ***(#Task2)***
Student 1	*" I was aware that point M is the intersection of two lines, but I was confused about how to determine its exact position. Should I use a coordinate system?"*
Student 2	*I used a vector approach to determine M, but I had trouble visualizing the positions of the lines in the cube. I tried drawing it, but it was hard to understand the spatial relationship."*

Twenty-one percent (21%) of metacognitive questions asked by the lecturers were related to the transfer of knowledge to real-world contexts. This finding indicated that questions related to the application of spatial geometry concepts in real life were rarely found in the classroom. In fact, knowing the connection between geometry concepts and their real-world applications can increase students' learning motivation. This finding also implies that the lecturers need to ask more real-world and case-based questions to help students connect geometry concepts to real situations. An example of these metacognitive questions can be seen below.

Lecturer	*"If you were an engineer, how would you apply the concept of point to plane distance in a construction project?* ***(#Task2)***

Metacognitive questions asked by the lecturers typically focused on raising students' awareness of geometry understanding and problem-solving strategies. Metacognitive questions related to knowledge transfer to real-world contexts had the lowest percentage, even though this aspect is important for building students' motivation and improving their critical thinking skills. In addition, technology-based approaches and group discussions can help students evaluate their geometric understanding and problem-solving strategies. Therefore, lecturers need to ask various types of metacognitive questions to ensure that students not only understand spatial geometry concepts but are also able to apply them in real-world situations. The section discusses the types of scaffolding provided by the lecturers in the learning process we observed. We provide links to the scaffolding questions asked by the lecturers at each meeting. A summary of the findings is presented in Table 5 and Figure 3.

Table 5. Metacognitive Questions and Scaffolding Provided by the Lecturers

Metacognitive questions	Scaffolding	Type of Scaffolding
*How did you know that the two calculated diagonals lie in three-dimensional space? **(#Task1)***	*Consider the diagonals BD and AH as two vectors in space. Can you imagine how they interact with each other?* *Let's use a model of a cuboid or a 3D image to visualize how the two diagonals meet in space*	Conceptual Scaffolding
	The first step is to identify the position vectors for each diagonal. Have you determined the coordinates of each point? *Use the dot product formula to calculate the angle between these two vectors. Do you remember the formula?*	Procedural Scaffolding
	Can you explain why we use the dot product to determine the angle between the two diagonals? *Is there another way to do this besides using the vector approach?*	Metacognitive Scaffolding
*How did you ensure that point M is truly the intersection of AF and BE? **(#Task2)***	*Let's use a cube diagram and mark the important points. Can you see how AF and BE intersect?* *We can visualize the intersection of two lines in 3D coordinates. Set the origin at one corner of the cube and observe how the other points are positioned.*	Conceptual Scaffolding
	The first step is to write the parametric equations for lines AF and BE. Have you found the direction vectors for each line? *Let's determine what the intersection point is by making the parametric equations equal to one another. What are the results? Do they turn out as you anticipated?*	Procedural Scaffolding
	What would happen to the distance between BH and MN if the point M moves in the opposite direction? *Is there another way to find the intersection point that does not use coordinates?*	Metacognitive Scaffolding

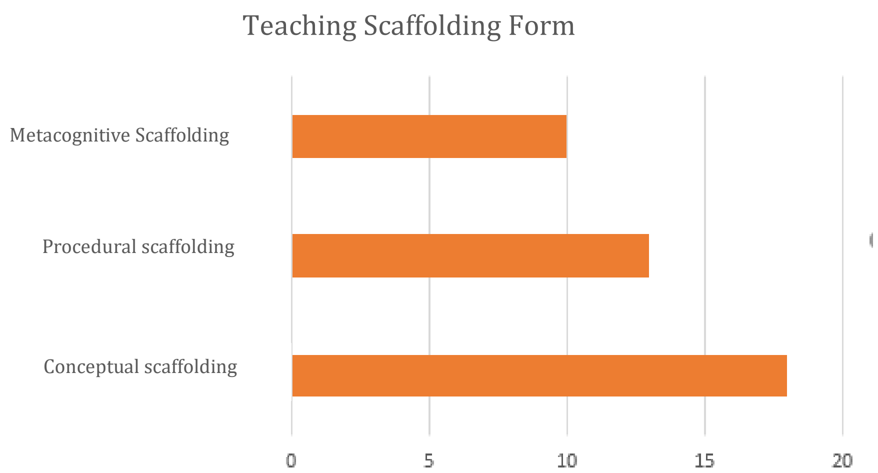

Figure 3. Scaffolding Used by The Lecturers

From the details found in Table 5 and Figure 3, the most common type of Conceptual Scaffolding recorded was 44%. This shows that lecturers mainly focused on assisting students to understand basic concepts in geometry (such as the relationship between points, lines, and planes) before engaging them in solving spatial geometry problems. A proper understanding of concepts will allow the learner to visualize and understand different objects in three-dimensional space. Additional support that lecturers provided to students involved assisting them in understanding the procedures used in solving geometric problems (around 31%). This appears to be intended for students who are lacking in knowing whether they can identify the procedures correctly or apply them appropriately when solving problems. To enhance the understanding of students, the lecturer could assist in presenting detailed processes with the aid of worksheets and assist with understanding the concepts in detail, so that those who might be lacking in understanding the processes involved in solving problems effectively can be helped with the resources available. Metacognitive scaffolding support was used least of all (around 24%). This implies that reflective questions intended to assist learners in understanding and critically evaluating the problem-solving process were least likely to be used by lecturers. This is good because studies have indicated that when students are prompted to regularly ask reflective questions and engage in reflecting on the problem-solving process itself, critical thinking is enhanced, and the problem-solving abilities of students are improved. Further details regarding the types of scaffolding provided throughout the instructional process are outlined in Table 6.

Table 6. Types of Scaffolding Provided throughout the Learning Process

Conceptual Scaffolding	Procedural Scaffolding	Metacognitive Scaffolding
• *"Before we calculate the distance from a point to a plane, let's first understand how to determine the normal vector of a plane."* • *"How can we determine the angle between two lines in space using the dot product?"* • *"Take a look at this example of cube visualization before calculating the length of its space diagonal"*	• *"First, find the vector connecting the two points. After that, use the vector distance formula to calculate its length."* • *"Before finding the distance between a point to a plane, write down the equation of the plane and make sure the given point does not lie in the plane."* • *"Use the Pythagorean theorem in the first step, then continue with the vector approach if needed."*	• *"Can you explain the steps you used to determine the intersection point of the two lines within the cube?"* • *"How can you be sure that your answer is correct?"* • *"Would applying an alternative method yield the same result? Explain your reasoning."*

Q2: To what extent is the integration of the flipped classroom and metacognitive scaffolding effective in geometry learning?

This section discusses the results of the descriptive analysis on students' pretest and post-test mean scores. According to Table 7, the mean pre-test scores for the three groups were relatively similar: Class A (Flipped Classroom with Metacognitive Scaffolding) scored 64.90**,** Class B (Conventional) scored 64.53**,** and Class C (Flipped Classroom without Scaffolding) scored 66.39**.** The scores speak for themselves: before anything happened to any of them, the classes were quite similar.

There was a nice widening of the disparity after the intervention: Class A averaged 79.06 on the post-test, Class C averaged 75.06, and Class B averaged 69.39. Classes had made the greatest gain between pre- and post-test in Class A, which would suggest that the metacognitive scaffolding work and flipping complemented each other better than the traditional approach or flipping alone in improving spatial ability.

When we check the results for normalized gain (N-gain), again the pattern continues. Class A takes first place with an N-gain of 0.386, followed by Class C with an N-gain of 0.241 and Class B with an N-gain of 0.119. We can thus assert that Class A experienced a quicker gain in learning achievements than the others. In spite of the flipped classroom method being implemented for Class C, it has failed to attain a high N-gain comparable to Class A.

The fact that Class B consistently performed lower in both post-test scores and N-Gains suggests that perhaps traditional teaching methods are not as conducive to enhancing spatial thinking. The overall results lend credibility to the understanding that metacognitive scaffolding in a flipped classroom environment is more conducive to student learning than either teaching method individually. They also serve to illustrate the importance of metacognitive scaffolding in understanding complex geometric concepts.

Table 7. Descriptive Analysis Results

Class	Mean Pre-Test	Mean Post-Test	Mean N-Gain	Std Error
A	64.90	79.06	0.386	0.82
B	64.53	69.39	0.11	0.85
C	66.39	75.06	0.24	0.91

As presented in Figure 4, the results of the descriptive analysis are arranged in an organized manner for better comprehension and explanation. From the line graph showing the comparison of pre-test and post-test results, it is clearly illustrated that students in Class A, Flipped Classroom with Metacognitive Scaffolding, rise further compared to students in Class B (Conventional) and Class C (Flipped Classroom Without Scaffolding). From the N-Gain line graph, it is likewise noted that Class A has the greatest growth, followed by Class C, and finally Class B with the lowest growth rate. It is noted that one of the most significant aspects here is the integration of metacognitive scaffolding, which is significant for producing better student results.

Figure 4. Visualization of the Descriptive Data

Afterward, we conducted inferential statistics to address the research questions. Normality test: We collected the post-test results for each class and carried out the Shapiro-Wilk test for each to test for normality. According to Table 8, we observed that all the p-values are greater than 0.05, indicating that the results after the post-test for all the classes are normally distributed. Therefore, a clear path is opened to the next tests, which are homogeneity tests and hypothesis tests.

Table 8. Normality Testing

Class	W Statistic	*p-value*	Normality
A (Flipped + Scaffolding)	0.97	*0.45*	Yes
B (Conventional)	0.96	*0.30*	Yes
C (Flipped without Scaffolding)	0.95	*0.25*	Yes

A test for checking the homogeneity of variances among the three groups of scores was performed, and it produced a result that showed that the variance among each of the three groups of scores is higher than 0.05 (see Table 9). This confirms the assumption that can be made for checking the equality of the mean scores.

Table 9. Homogeneity Test Result

Levene Statistic	*P-Value*	Homogeneous
0.15	*0.86*	Yes

Inferential test results were used to examine the scores of the pre-test and post-test among the three groups. From the test results, it was evident that there was a marked difference between Class A and Class B. This indicated that the use of the Flipped Classroom with Metacognitive Scaffolding yielded a better outcome than the Conventional method. It was also evident that there were diverging scores between Class A and Class C, which meant that the inclusion of Metacognitive Scaffolding yielded an additional learning advantage. Finally, from the inferential test, it was also evident that the Flipped Classroom method, irrespective of the inclusion of Metacognitive Scaffolding, yielded a higher outcome than the Conventional method, as shown in Table 10 Comparison Test.

Table 10. Comparison Test

Comparison	t-stat	p-value	Significance (p<0.05)
A Vs B	7.22	0.001	Yes
A Vs C	3.18	0.004	Yes
B Vs C	-4.02	0.001	Yes

In order to find out which of the groups was significantly different from one another, the Tukey HSD Post-Hoc Test was carried out. From a review of the Table 11, Comparison Class Score, a number of pertinent issues can be deduced. First and foremost, the Class A (Flipped Classroom with Metacognitive Scaffolding) had a significantly higher average than Class B (Conventional) and Class C (Flipped Classroom without Scaffolding). In addition, Class C scored higher than Class B but fell far short of Class A. From this data, it is unequivocally clear that the best strategy to increase students' spatial ability is to employ a combination of a Flipped Classroom and Metacognitive Scaffolding.

Table 11. Comparison Class Score

Group 1	Group 2	Mean Diff	p-value	Significant
A (Flipped + Scaffolding Metacognitive)	B (Conventional)	-9.67	0.001	Yes
A (Flipped + Scaffolding Metacognitive)	C (Flipped without Scaffolding Metacognitive	-3.99	0.004	Yes
B (Conventional)	C (Flipped without Scaffolding Metacognitive	5.67	0.001	Yes

The findings showed that a combination of the Flipped Classroom with Metacognitive Scaffolding, Class A, can be considered the most powerful arrangement to enhance students' understanding of spatial geometry. The setting of the Flipped Classroom without Scaffolding, Class C, outperformed Class B, the conventional method, though it did not perform like the combination in Class A. From the observations, it can be seen that metacognitive scaffolding could be one of the critical contributors that enhance students' spatial reasoning when learning about geometry.

There are four key points worth further discussion. First, analyzing the students' needs in comprehending geometric concepts highlights specific needs in developing spatial reasoning. The study also determines what kind of metacognitive questions instructors use to foster students' spatial geometry skills. Finally, it considers what type(s) of scaffolding best facilitate students in understanding spatial geometry concepts. The results reveal that students at the university level require strengthening in four areas: (1) understanding three-dimensional coordinates, (2) mastering geometric transformations, (3) being able to visualize spatial figures, and (4) comprehending the relations between relevant geometric elements, such as angles and distances. It appears that a major hurdle to understanding three-dimensional geometry is the students' experience of difficulty in creating mental spatial representations. If students are not able to imagine the locations of points or orientations of lines in space, it becomes difficult to link algebraic representations to the appropriate spatial structures (Cromley et al., 2017; Harris et al., 2023; Ramful et al., 2017). Students who struggle to visualize the positions of points or the directions of lines in space often face challenges in linking algebraic representations to their corresponding spatial structures (Haciomeroglu & LaVenia, 2017; Harris et al., 2023; Pitta-Pantazi et al., 2020). To address these issues, instructional strategies should incorporate interactive visual tools, such as GeoGebra or Augmented Reality (AR), as recommended by previous studies (Xin, 2019). To address these difficulties, instructional methods need to utilize interactive visual tools, GeoGebra, AR, and other similar tools, as previous research has suggested. Such tools promote linking symbolic abstraction and spatial visualization for strengthening conceptual understanding of spatial geometry.

Similarly, in scaffolding, the research highlights that teachers favor the use of conceptual and procedural scaffolding; in contrast, metacognitive scaffolding occurs very rarely. This is important, as metacognitive scaffolding is considered vital for developing learner autonomy and for enhancing the efficiency of students' learning strategies (Tondorf & Prediger, 2022). The metacognitive scaffolding can assist university students to think about their own thinking processes and select an effective learning strategy so that they can successfully deal with complex spatial problem-solving tasks (Hartmann et al., 2024; Yimer & Ellerton, 2010). This lack of metacognitive scaffolding exposes a critical lack of training opportunities for the students to reflect on their learning experience.

In the context of scaffolding, researchers saw that most of the scaffolding occurred conceptually and procedurally, whereas metacognitive scaffolding occurred only a little. This is noteworthy, as metacognitive scaffolding plays a crucial role in fostering learner autonomy and enhancing the effectiveness of students' learning strategies (Chang et al., 2016; Hein & Prediger, 2024; Kosko, 2020; Speer & Wagner, 2009). Metacognitive scaffolding is critical to enabling learners to take charge of their learning and to make students' self-regulation of their own study more effective. Where metacognitive scaffolding is evident, university students can engage their cognitive skills in assessing their ways of thinking, selecting appropriate approaches, and developing more flexible ways of thinking when they encounter difficult spatial problems (Harris et al., 2023; Ramful et al., 2017). Limited metacognitive scaffolding suggests that perhaps students are not being sufficiently encouraged to develop critical ways of reflecting on their learning experiences. It is also important to recognize the blending of students' conceptual and visual needs with the opportunity for metacognitive questioning and student scaffolding. Simply presenting information or teaching them how to do something may not suffice; a process of guidance through the thinking process ought to be applied for them to understand what they are to do and why a particular approach applies to a particular endeavor.

This study examines the process of geometry learning in classes using a flipped class and metacognitive scaffolding. Findings show that this approach results in an additional productive improvement in spatial thinking, unlike other forms of instruction. This increase in spatial thinking can also be justified by constructivism, which advocates for meaningful learning to occur when learners are encouraged to develop their own concepts by creating learning experiences on their own, even though it is done in a guided manner (Sevinc & Lesh, 2022). This occurs in a flipped class since learners are initially challenged to learn on their own and become more independent and knowledgeable. There is also the element of a plan of thinking using metacognitive scaffolding to ensure the experimentation carried out by learners is more meaningful and cognitively related. The effective application of scaffolding has to fit the predominance of learners' cognitive and metacognitive demands, allowing change along the learning path (Cevikbas & Kaiser, 2020; Kosko, 2020).

Past research demonstrated a limited effect of implementing a flipped classroom strategy on a student's conceptual understanding of math. Apart from that, research indicated a stronger probability of engaging students through a flipped classroom, while improving time on task is still unclear, especially where a reflective support strategy is absent (Cronhjort et al., 2018; Fredriksen, 2021). From the current research, a conclusion can be drawn that a flipped classroom strategy is associated with enhanced understanding of spatial geometry, provided that the strategy is implemented together with a metacognitive support strategy. In other words, the impact of the flipped classroom strategy might change, depending more on its effective support than its form.

The findings below elucidate why previous scaffolding research in mathematics education often seems to conflict. When research limits itself to either procedural or conceptual scaffolding, we frequently find improved task completion with limited transfer and independence (Nagle et al., 2019). In contrast, this study suggests that metacognitive scaffolding encourages students to examine their own reasoning and contemplate relationships in space to develop a deeper conceptual understanding. The results, therefore, explain why scaffolding-based interventions do not always lead to the same results when different studies are compared.

From a cognitive point of view, the gains we observe can be understood in terms of how learners coordinate multiple representations. Spatial geometry requires students to integrate symbolic formulas, visual diagrams, and internal spatial models. The flipped classroom design accommodates the creation of these representations in pre-class work, and in-class metacognitive scaffolding encourages students to monitor, adjust, and justify the products they have generated. This coordinated process may be the reason that students in the integrated condition outperform students in an unscaffolded flipped classroom design, where students often lack structured opportunities to regulate and refine their spatial reasoning. The sizes of those effects suggest that, not only are we seeing significant changes, but also that they are significant and meaningful changes. So, it's not just significant, it's significant, and it has some real meaning around it. Given the complexity of spatial geometry, and given that it's a brief intervention, it also has pretty real-world applicability and impact.

The overall findings, then, are to some extent indicative that what promotes effective spatial geometry instruction is not exactly the tech format, but the degree to which the instruction is organized. This particular piece of research contributes to the overall literature by offering a clearer understanding of the means by which metacognitive scaffolding acts as a key bridge between the flipped classroom and organized, thought-strategically effective learning. This accordingly explains the mixed results found in the preponderance of flipped classroom research and sharpens the theory involving learner-centered education.

The basic idea in this context is clearly related to learner-centered design. In such a system, it is clear that metacognitive scaffolding functions as the key connection to be made between where learners are and where they want to be in terms of higher-level cognitive capabilities (Hartmann et al., 2024; Trapman et al., 2018). If learners are helped to observe, monitor, and even think about their particular strategies, they can easily transition from being 'information receivers' to being 'drivers of their own development' (Rach & Ufer, 2020). This, in turn, can be extremely important concerning something like spatial geometry, considering that learners have to be able to visualize, analyze, and eventually decide based on complexly related spaces (Montenegro et al., 2018; Tall, 2008; Yao, 2022).

This inclusion also ensures that time management plays a major part in the class sessions; more space for metacognitive activity can be provided through dialogue and feedback sessions. This approach is more productive than relying on conventional lectures. When reflective practices are also incorporated into the flipped learning approach, more effective understanding and development of problem-solving abilities may be achieved (Cevikbas & Kaiser, 2020; Shaughnessy et al., 2021). Thus, the flipped approach to learning and teaching, augmented by metacognitive practices, not just proves to be more productive than conventional ones for improving spatial ability but also provides a more effective and reflective approach to learning by being more autonomous and meaningful—a factor that becomes highly necessary for geometry learning in the digital world.

LIMITATIONS

It must be noted, however, that there are some aspects of the study to be improved in further research. For instance, the present study was able to cover only two major topics in spatial geometry. Moreover, it was noticed that while lecturers and students participated in this survey, they came from a particular institution, which may introduce some specific features reflecting a particular ecological context. It is also to be noted that while this study employed both qualitative and quantitative methods, there was no attempt to analyze development patterns in the metacognitive abilities of students using longitudinal approaches.

From what is found and with regard to what was not confirmed, it is suggested that two general ways of further research have been identified. First, it is noted that the issues explored and focused on in this research were related to angles and distances of elements in cubes and cuboids. This way, further research should include more complex items, such as geometry about curved surfaces, transformation geometry, and other issues that require better visualization skills. Second, it is noted that the research is focused on short-term practice. Further research is needed to trace and explain how students' metacognitive skills change over time and how students' spatial understanding is stored and transferred.

CONCLUSION

However, the research aims to highlight the importance of geometry, as a branch of mathematics, while employing a flipped classroom integrated with metacognitive scaffolding techniques. According to the research result, there is a clear indication of the need for further development in certain important aspects, such as understanding three-dimensional coordinates, gaining knowledge about geometric transformations, developing spatial imagination, and grasping the relationship between elements of a geometric shape, such as its angles. It emphasizes the ideal practice of teaching geometry, which is not just about procedures, as it is lacking in filling the conceptual gaps involved in these important aspects. Students are not only able to understand the main ideas but are also able to improve their metacognitive skills through reflection, evaluation, and decision-making in solving problems via a combination of a flipped classroom structure and metacognitive scaffolding techniques, which is more effective than using traditional teaching approaches, as well as the flipped classroom approach only.

What this study shows is that a flipped classroom can already lead to better learning results than traditional teaching, even without metacognitive scaffolding. The reason seems to be that seeing visual and digital materials in advance of the class helps students build initial spatial ideas beforehand, after which they tackle in-class problems. But when you combine it with metacognitive scaffolding, the approach gets even stronger. By explicitly guiding students to monitor and adjust their thinking and to fine-tune their spatial reasoning, the model becomes more effective. Overall, in this study, the flipped classroom paired with metacognitive scaffolding was found to yield greater improvements than conventional instruction alone and the flipped approach sans scaffolding. Independent study is combined with cognitive reinforcement and guided reflection. In light of these findings, the geometry instructions should include reflection-based learning strategies, particularly concerning topics beyond advanced visualization and spatial reasoning. However, not all uses of metacognitive scaffolding are effective; for example, it can perform poorly when the scaffolding is too general, or when it is not coherently matched to the knowledge which the students already possess, or when used in a manner limiting students from developing their own strategies. These subtleties indicate that adaptive, context-sensitive scaffolding design is necessary rather than a one-size-fits-all collection of metacognitive prompts.

AUTHOR CONTRIBUTIONS

AA played the leading role in the overall concept and design of the study, including developing the theoretical underpinnings and linking the flipped classroom to metacognitive scaffolding. Sutamrin Sutamrin contributed to the implementation of the study, including data collection in the classroom and drafting of the results and discussion. HI contributed to validating the research tools, understanding the research output, and added a critical review component to make the research more plausible in terms of theory and methodology. NN contributed to data curation, organizing the research, and creating visual representations of the research data in terms of tables and figures. MAN contributed to creating the literature review, facilitating the use of digital platforms, creating statistical tools, drafting aspects of the manuscript, creating contrasting backgrounds for research, and determining research methodologies. MI contributed to developing the research theory, improving the manuscript readability, and creating a critical review component for coherence in theory.

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