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On the normal product and modular product of product fuzzy graphs with applications to defense communication networks

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Abstract

Background: This study focuses on the development of new operations of product fuzzy graphs, specifically the normal product and modular product operations. The normal product of product fuzzy graphs is used to build infrastructure with maximum path redundancy. Meanwhile, the modular product operation of product fuzzy graphs is implemented to optimize the alignment of operational processes and information security.
Aims: This study aims to define the normal product and modular product of product fuzzy graphs, define the properties of the normal product and modular product of product fuzzy graphs, and to apply to defense communication networks.
Method: The research stages consist of formulating the definition of the normal product and modular product of product fuzzy graphs along with its properties and applying to defense communication networks.
Results: The results of the study show that the normal product and modular product of product fuzzy graph is fuzzy graph, the normal product and modular product of strong product fuzzy graph is strong, consequently, the normal product and modular product of complete product fuzzy graph is strong.
Conclusion: The application of the normal product and modular product of product fuzzy graph can produce a more compact defense communication network.

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INTRODUCTION

In defense systems, the integrity of strategic information is often threatened by security breaches in communication network infrastructure. The relationships between nodes in a communication network often involve uncertainties that cannot be described by classical graphs. To address this limitation, the concept of fuzzy graphs was introduced as a generalization of classical graphs capable of accommodating data ambiguity. In both theoretical and applied developments, various variations of fuzzy graphs have been proposed, such as product fuzzy graphs (Ramaswamy & Poornima, 2009), interval valued fuzzy graph (Akram & Dudek, 2011), pythagorean fuzzy graphs (Akram et al., 2018), hesitant fuzzy graphs (Javaid et al., 2020), cubic fuzzy picture graphs (Khatun et al., 2024) and certain operation (Khatun et al., 2025), and plithogenic fuzzy graphs (Azeem et al., 2025). Other variants of fuzzy graphs include neutrosophic cubic graphs (Gulistan et al., 2018), semi fuzzy graph (Al-Hawary & Hashim, 2025), and complex vague graphs (Zeng et al., 2020), and the edge connectivity of semi-strong product graphs (Wang & Ren, 2025), and the concept of level graphs in interval-valuad fuzzy graph (Rao et al., 2023).

The application of fuzzy graphs has expanded rapidly across various strategic sectors due to their ability to handle uncertain and ambiguous data. In the infrastructure and transportation management sector, fuzzy graphs are effectively used in managing traffic flow networks (Ahmad et al., 2023), as well as intelligent traffic control management (Nagarajan et al., 2019). In other fields, fuzzy graphs are used for modeling drug molecules (Wang et al., 2025), fuzzy graph to assist in medical data processing (Meenakshi et al., 2026), and to help model the global spread of infectious diseases (Sultana et al., 2023), as well as analyzing the impact of pollution based on the degree of node connectivity (Al-Omeri et al., 2025). Additionally, these applications span the textile industry (Nazeer et al., 2021), corporate merger processes (Nawaz & Ahmad, 2023), and the identification of dominant figures in social media interactions (Alqahtani et al., 2024). Furthermore, operations on fuzzy graphs help determine complex optimal routes (Raja et al., 2024), as well as evaluate network integrity in mass transit systems such as metro rail (Sankar et al., 2025). In the field of telecommunications, connectivity number analysis on fuzzy graphs is crucial for maintaining service stability in cellular communication networks (Rezayi et al., 2025), complex neutrosophic hesitant fuzzy graph is used to solve a problem related to cellular network (Poonia & Bhardwaj, 2023), and geodetic domination integrity in fuzzy graphs (Ganesan et al., 2023). The modeling of networks studies on multidimensional fuzzy graphs that explore aspects of connectivity (Josen & John, 2026a) and dominance (Josen & John, 2026b).

The product fuzzy graph is essentially an extension of the fuzzy graph achieved by replacing the minimum operator in the fuzzy graph with the product operator (Ramaswamy & Poornima, 2009). Other results related to product fuzzy graphs include properties of product fuzzy graphs (Al-Hawary & Hourani, 2016), intuitionistic product fuzzy graphs (Al-Hawary & Hourani, 2017), and the global domination number in product fuzzy graphs (Shubatah & Haifa, 2023). The development of the properties of the product fuzzy graph heavily depends on algebraic operations capable of preserving the integrity of the relationships between elements (Dogra, 2015). In addition, there is also the corona product of product fuzzy graphs (Al-Hawary, 2024), γ-product of product fuzzy graphs (Al-Hawary, 2025a), and α-product of anti product fuzzy graphs (Al-Hawary, 2025b). To date, several studies have explored the effective use of the cartesian product in modeling orthogonal relationships among network elements (Dhanya et al., 2025), cartesian product operation of product fuzzy graph (Firmansah et al., 2025), and tensor product of product fuzzy graphs (Firmansah, 2025). This approach extends the security allocation concept proposed into an integrated communication framework (Nair & Sunitha, 2024), modeling cybercrime issues (Ahmad et al., 2023), and analyzing crime patterns on railway networks (Islam & Pal, 2023).

This study addresses this gap by defining the normal product and modular product of product fuzzy graphs. The use of the normal product operation offers advantages in path redundancy accuracy that are not possessed by the cartesian product and tensor product. Meanwhile, the modular product operation of product fuzzy graphs is implemented to optimize the alignment of operational processes and information security. Theoretically, this study aims to define the normal product and modular product of product fuzzy graphs along with their structural properties. This includes proving that the normal product and modular product of product fuzzy graphs is a fuzzy graph, the normal product and modular product of strong product fuzzy graph is a strong, and the normal product and modular product of complete product fuzzy graph is a strong. This study implements the normal product and modular product of product fuzzy graphs to generate a more compact design for defense communication networks.

METHOD

This section formalizes the mathematical method used to 1) define the normal product and modular product of product fuzzy graph, 2) prove that the normal product and modular product of product fuzzy graph admits an fuzzy graph, 3) prove that the normal product and modular product of strong product fuzzy graph is a strong, 4) prove that the normal product and modular product of complete product fuzzy graph is a strong, and 5) the application of the normal product and modular product of product fuzzy graph for defense communication network.

Preliminaries and Notation

Definition 1. (Rosenfeld, 1975)
Let $G^* = (V, E)$ be a graph, α be a fuzzy subset of V and β fuzzy subset of $V \times V$. We call $G = (\alpha, \beta)$ fuzzy graph where $\alpha: V \to [0,1]$ is a fuzzy subset and $\beta: V \times V \to [0,1]$ is a fuzzy relation on α such that $\beta(xy) \le \alpha(x) \wedge \alpha(y)$ for all $x, y \in V$, where $\wedge$ stands for minimum.

Definition 2. (Ramaswamy & Poornima, 2009)
Let $G^* = (V, E)$ be a graph, α be a fuzzy subset of V and β fuzzy subset of $V \times V$. We call $G = (\alpha, \beta)$ product fuzzy graph where $\alpha: V \to [0,1]$ is a fuzzy subset and $\beta: V \times V \to [0,1]$ is a fuzzy relation on α such that $\beta(xy) \le \alpha(x) \times \alpha(y)$ for all $x, y \in V$, where $\times$ stands for product.

Definition 3. (Ramaswamy & Poornima, 2009)
A product fuzzy graph $G = (\alpha, \beta)$ with $G^* = (V, E)$ is said to be strong if $\beta(xy) = \alpha(x) \times \alpha(y)$ for all $(x, y) \in E$.

Definition 4. (Ramaswamy & Poornima, 2009)
A product fuzzy graph $G = (\alpha, \beta)$ with $G^* = (V, E)$ is said to be complete if $\beta(xy) = \alpha(x) \times \alpha(y)$ for all $x, y \in$ V.

Definition 5. (Firmansah et al., 2025),
Cartesian product of product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ and $V_1 \cap V_2 \neq \emptyset$ is a graph $G_1 \times G_2 = (\alpha_1 \times \alpha_2, \beta_1 \times \beta_2)$ with $G^* = (V_1 \times V_2, E_1 \times E_2)$ where $V_1 \times V_2 = \{(u_1, v_1) | u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and
$E_1 \times E_2 = \{(u_1, v_1)(u_2, v_2) | u_1 = u_2, v_1v_2 \in E_2 \text{ or } u_1u_2 \in E_1, v_1 = v_2\}$
$(\alpha_1 \times \alpha_2)(u_1, v_1) = \alpha_1(u_1) \times \alpha_2(v_1)$ for all $(u_1, v_1) \in V_1 \times V_2$

$$(\beta_1 \times \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \begin{cases} \big(\alpha_1(u_1)\big)^2 \times \beta_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \\ \beta_1(u_1u_2) \times \big(\alpha_2(v_1)\big)^2, \text{if } u_1u_2 \in E_1, v_1 = v_2 \end{cases}$$

Definition 6. (Firmansah, 2025)
Tensor product of product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ and $V_1 \cap V_2 \neq \emptyset$ is a graph $G_1 \otimes G_2 = (\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ with $G^*: (V_1 \otimes V_2, E_1 \otimes E_2)$ where $V_1 \otimes V_2 = \{(u_1, v_1) | u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and
$E_1 \otimes E_2 = \{(u_1, v_1)(u_2, v_2) | u_1u_2 \in E_1, v_1v_2 \in E_2\}$
$(\alpha_1 \otimes \alpha_2)(u_1, v_1) = \alpha_1(u_1) \times \alpha_2(v_1)$ for all $(u_1, v_1) \in V_1 \otimes V_2$

$$(\beta_1 \otimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \beta_1(u_1u_2) \times \beta_2(v_1v_2), \text{if } u_1u_2 \in E_1, v_1v_2 \in E_2$$

Definition 7. (Dogra, 2015)
Normal product of fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ and $V_1 \cap V_2 \neq \emptyset$, is a graph $G_1 \boxtimes G_2 = (\alpha_1 \boxtimes \alpha_2, \beta_1 \boxtimes \beta_2)$ with $G^* = (V_1 \boxtimes V_2, E_1 \boxtimes E_2)$ where $V_1 \boxtimes V_2 = \{(u_1, v_1) | u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and

$$E_1 \boxtimes E_2 = \left\{(u_1, v_1)(u_2, v_2) \middle| \begin{matrix} u_1 = u_2, v_1v_2 \in E_2 \text{ or } u_1u_2 \in E_1, v_1 = v_2 \\ \text{or } u_1u_2 \in E_1, v_1v_2 \in E_2 \end{matrix} \right\}$$

$(\alpha_1 \boxtimes \alpha_2)(u_1, v_1) = \alpha_1(u_1) \wedge \alpha_2(v_1)$ for all $(u_1, v_1) \in V_1 \boxtimes V_2$

$$(\beta_1 \boxtimes \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \begin{cases} \alpha_1(u_1) \wedge \beta_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \\ \beta_1(u_1u_2) \wedge \alpha_2(v_1), \text{if } u_1u_2 \in E_1, v_1 = v_2 \\ \beta_1(u_1u_2) \wedge \beta_2(v_1v_2), \text{if } u_1u_2 \in E_1, v_1v_2 \in E_2 \end{cases}$$

Definition 8. (Dogra, 2015)
Modular product of fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ and $V_1 \cap V_2 \neq \emptyset$ is a graph $G_1 \odot G_2 = (\alpha_1 \odot \alpha_2, \beta_1 \odot \beta_2)$ atas $G^* = (V_1 \odot V_2, E_1 \odot E_2)$ where $V_1 \odot V_2 = \{(u_1,v_1) | u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and

$$E_1 \odot E_2 = \left\{(u_1,v_1)(u_2,v_2) \middle| \begin{matrix} u_1u_2 \in E_1, v_1v_2 \in E_2 \\ \text{or } u_1u_2 \notin E_1, v_1v_2 \notin E_2 \end{matrix} \right\}$$

$(\alpha_1 \odot \alpha_2)(u_1,v_1) = \alpha_1(u_1) \wedge \alpha_2(v_1)$ for all $(u_1,v_1) \in V_1 \odot V_2$

$$(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \begin{cases} \beta_1(u_1u_2) \wedge \beta_2(v_1v_2), if\ u_1u_2 \in E_1, v_1v_2 \in E_2 \\ \alpha_1(u_1) \wedge \alpha_1(u_2) \wedge \alpha_2(v_1) \wedge \alpha_2(v_2)\ if\ u_1u_2 \notin E_1, v_1v_2 \notin E_2 \end{cases}$$

RESULTS AND DISCUSSION

Normal Product of Product Fuzzy Graph

The following provides the normal product definition of product fuzzy graphs along with its properties.

Definition 9.
Normal product of product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ and $V_1 \cap V_2 \neq \emptyset$ is a graph $G_1 \boxtimes G_2 = (\alpha_1 \boxtimes \alpha_2, \beta_1 \boxtimes \beta_2)$ with $G^* = (V_1 \boxtimes V_2, E_1 \boxtimes E_2)$ where $V_1 \boxtimes V_2 = \{(u_1,v_1) | u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and

$$E_1 \boxtimes E_2 = \left\{(u_1,v_1)(u_2,v_2) \middle| \begin{matrix} u_1 = u_2, v_1v_2 \in E_2 \text{ or } u_1u_2 \in E_1, v_1 = v_2 \\ \text{or } u_1u_2 \in E_1, v_1v_2 \in E_2 \end{matrix} \right\}$$

$(\alpha_1 \boxtimes \alpha_2)(u_1,v_1) = \alpha_1(u_1) \times \alpha_2(v_1)$ for all $(u_1,v_1) \in V_1 \boxtimes V_2$

$$(\beta_1 \boxtimes \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \begin{cases} \big(\alpha_1(u_1)\big)^2 \times \beta_2(v_1v_2), \text{if } u_1 = u_2, v_1v_2 \in E_2 \\ \beta_1(u_1u_2) \times \big(\alpha_2(v_1)\big)^2, \text{if } u_1u_2 \in E_1, v_1 = v_2 \\ \beta_1(u_1u_2) \times \beta_2(v_1v_2), \text{if } u_1u_2 \in E_1, v_1v_2 \in E_2 \end{cases}$$

Example 1.
Let product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $\alpha_1(u_1) = 0.5$, $\alpha_1(u_2) = 0.6$, $\alpha_1(u_3) = 0.4$, $\beta_1(u_1u_2) = 0.2$, $\beta_1(u_2u_3) = 0.1$ and product fuzzy graph $G_2 = (\alpha_2, \beta_2)$ with $\alpha_2(v_1) = 0.3$, $\alpha_2(v_2) = 0.6$, $\beta_2(v_1v_2) = 0.1$. In Figure 1, graph $G_1 \boxtimes G_2$ is obtained as follows.

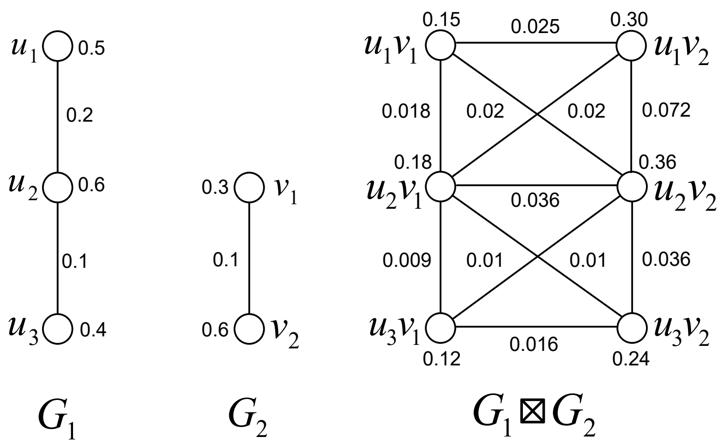

Figure 1. The normal product of product fuzzy graphs $G_1 \boxtimes G_2$.

Theorem 10.
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ are product fuzzy graphs, then the normal product $G_1 \boxtimes G_2 = (\sigma_1 \boxtimes \sigma_2, \mu_1 \boxtimes \mu_2)$ is a fuzzy graph.

Proof:
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ be a product fuzzy graph, then it satisfies

$\beta_1(u_1u_2) \le \alpha_1(u_1) \times \alpha_1(u_2) \le \alpha_1(u_1) \wedge \alpha_1(u_2)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ be a product fuzzy graph, then it satisfies $\beta_2(v_1v_2) \le \alpha_2(v_1) \times \alpha_2(v_2) \le \alpha_2(v_1) \wedge \alpha_2(v_2)$. For the verification process, we divide it into three cases, namely
Case 1
For $u_1 = u_2, v_1v_2 \in E_2$
$(\beta_1 \boxtimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \big(\alpha_1(u_1)\big)^2 \times \beta_2(v_1v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_1) \times \beta_2(v_1v_2)$, because $u_1 = u_2$ is obtained
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \beta_2(v_1v_2)$
$\le \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$= \big(\alpha_1(u_1) \times \alpha_2(v_1)\big) \times \big(\alpha_1(u_2) \times \alpha_2(v_2)\big)$
$= (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \times (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
$\le (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \wedge (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
Case 2
For $u_1u_2 \in E_1, v_1 = v_2$
$(\beta_1 \boxtimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \beta_1(u_1u_2) \times \big(\alpha_2(v_1)\big)^2$
$= \beta_1(u_1u_2) \times \alpha_2(v_1) \times \alpha_2(v_1)$, because $v_1 = v_2$ is obtained
$= \beta_1(u_1u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$\le \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$= \big(\alpha_1(u_1) \times \alpha_2(v_1)\big) \times \big(\alpha_1(u_2) \times \alpha_2(v_2)\big)$
$= (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \times (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
$\le (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \wedge (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
Case 3
For $u_1u_2 \in E_1, v_1v_2 \in E_2$
$(\beta_1 \boxtimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \beta_1(u_1u_2) \times \beta_2(v_1v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$\le (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \wedge (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
Based on case 1, case 2, and case 3 we obtain $(\beta_1 \boxtimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) \le (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \wedge (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$ such that the normal product of the product fuzzy graph $G_1 \boxtimes G_2 = (\alpha_1 \boxtimes \alpha_2, \beta_1 \boxtimes \beta_2)$ is a fuzzy graph.

Theorem 11.
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ are strong product fuzzy graphs, then the normal product $G_1 \boxtimes G_2 = (\alpha_1 \boxtimes \alpha_2, \beta_1 \boxtimes \beta_2)$ is a strong.

Proof:
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ be a strong product fuzzy graph, then it satisfies $\beta_1(u_1u_2) = \alpha_1(u_1) \times \alpha_1(u_2)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ is a strong product fuzzy graph, then it satisfies $\beta_2(v_1v_2) = \alpha_2(v_1) \times \alpha_2(v_2)$. For the verification process, we divide it into three cases, namely
Case 1
For $u_1 = u_2, v_1v_2 \in E_2$
$(\beta_1 \boxtimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \big(\alpha_1(u_1)\big)^2 \times \beta_2(v_1v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_1) \times \beta_2(v_1v_2)$, because $u_1 = u_2$ is obtained
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \beta_2(v_1v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$= \big(\alpha_1(u_1) \times \alpha_2(v_1)\big) \times \big(\alpha_1(u_2) \times \alpha_2(v_2)\big)$
$= (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \times (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
Case 2
For $u_1u_2 \in E_1, v_1 = v_2$
$(\beta_1 \boxtimes \beta_2)\big((u_1, v_1)(u_2, v_2)\big) = \beta_1(u_1u_2) \times \big(\alpha_2(v_1)\big)^2$
$= \beta_1(u_1u_2) \times \alpha_2(v_1) \times \alpha_2(v_1)$, because $v_1 = v_2$ is obtained
$= \beta_1(u_1u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$

$= (\alpha_1(u_1) \times \alpha_2(v_1)) \times (\alpha_1(u_2) \times \alpha_2(v_2))$
$= (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \times (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
Case 3
For $u_1u_2 \in E_1, v_1v_2 \in E_2$
$(\beta_1 \boxtimes \beta_2)((u_1, v_1)(u_2, v_2)) = \beta_1(u_1u_2) \times \beta_2(v_1v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$= (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \times (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$
Based on case 1, case 2, and case 3 we obtain $(\beta_1 \boxtimes \beta_2)((u_1, v_1)(u_2, v_2)) = (\alpha_1 \boxtimes \alpha_2)(u_1, v_1) \times (\alpha_1 \boxtimes \alpha_2)(u_2, v_2)$ such that the normal product of the strong product fuzzy graph $G_1 \boxtimes G_2 = (\alpha_1 \boxtimes \alpha_2, \beta_1 \boxtimes \beta_2)$ is a strong.

Corollary 12.
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ are complete product fuzzy graphs, then the normal product $G_1 \boxtimes G_2 = (\alpha_1 \boxtimes \alpha_2, \beta_1 \boxtimes \beta_2)$ is a strong.

Example 2.
Let strong product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $\alpha_1(u_1) = 0.5$, $\alpha_1(u_2) = 0.6$, $\alpha_1(u_3) = 0.4$, $\beta_1(u_1u_2) = 0.3$, $\beta_1(u_2u_3) = 0.24$ and strong product fuzzy graph $G_2 = (\alpha_2, \beta_2)$ with $\alpha_2(v_1) = 0.3$, $\alpha_2(v_2) = 0.6$, $\beta_2(v_1v_2) = 0.18$. In Figure 2, graph $G_1 \boxtimes G_2$ is obtained as follows.

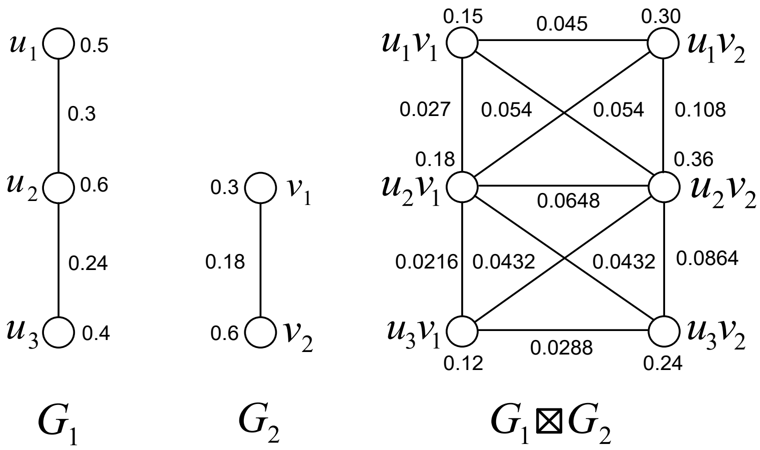

Figure 2. The normal product of strong product fuzzy graphs $G_1 \boxtimes G_2$.

Based on Definition 9, it follows that the normal product of product fuzzy graph combines the orthogonal relation of the cartesian product and the diagonal relation of the tensor product, resulting in $E(G_1 \boxtimes G_2) = E(G_1 \times G_2) \cup E(G_1 \otimes G_2)$. Furthermore, the properties of the normal product of product fuzzy graphs are established: the normal product of product fuzzy graph is a fuzzy graph as stated in Theorem 10; the normal product of strong product fuzzy graph is a strong as stated in Theorem 11; and the normal product of complete product fuzzy graph is a strong as stated in Corollary 12. These results are consistent with previous research based on (Al-Hawary & Hourani, 2016), (Firmansah et al., 2025), and (Firmansah, 2025).

Modular Product of Product Fuzzy Graph
The following provides the modular product definition of product fuzzy graphs along with its properties.
Definition 13.
Modular product of product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ and $V_1 \cap V_2 \neq \emptyset$ is a graph $G_1 \odot G_2 = (\alpha_1 \odot \alpha_2, \beta_1 \odot \beta_2)$ with $G^* = (V_1 \odot V_2, E_1 \odot E_2)$ where $V_1 \odot V_2 = \{(u_1, v_1) | u_1 \in V_1 \text{ and } v_1 \in V_2\}$ and

$$E_1 \odot E_2 = \left\{(u_1, v_1)(u_2, v_2) \middle| \begin{matrix} u_1u_2 \in E_1, v_1v_2 \in E_2 \\ \text{or } u_1u_2 \notin E_1, v_1v_2 \notin E_2 \end{matrix} \right\}$$

$(\alpha_1 \odot \alpha_2)(u_1, v_1) = \alpha_1(u_1) \times \alpha_2(v_1)$ for all $(u_1, v_1) \in V_1 \odot V_2$

$$(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \begin{cases} \beta_1(u_1u_2) \times \beta_2(v_1v_2), if\ u_1u_2 \in E_1, v_1v_2 \in E_2 \\ \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)\ if\ u_1u_2 \notin E_1, v_1v_2 \notin E_2 \end{cases}$$

Example 3.

Let product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $\alpha_1(u_1) = 0.2$, $\alpha_1(u_2) = 0.3$, $\alpha_1(u_3) = 0.4$, $\beta_1(u_1u_2) = 0.1$, $\beta_1(u_2u_3) = 0.3, \beta_1(u_1u_3) = 0.2$ and product fuzzy graph $G_2 = (\alpha_2, \beta_2)$ with $\alpha_2(v_1) = 0.5$, $\alpha_2(v_2) = 0.6, \beta_2(v_1v_2) = 0.6$. In Figure 3, graph $G_1 \odot G_2$ is obtained as follows.

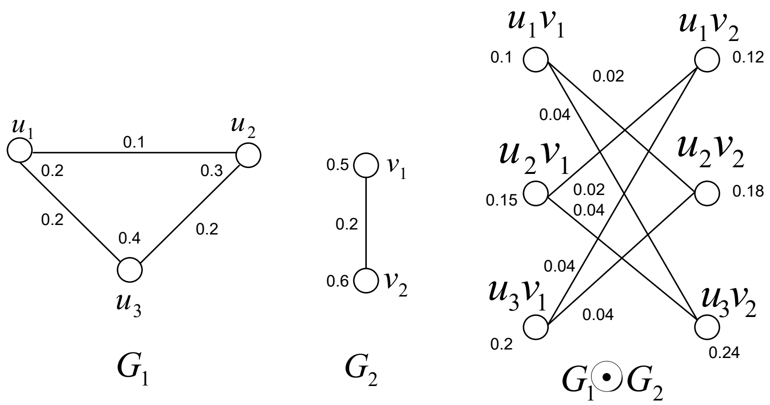

Figure 3. The modular product of product fuzzy graphs $G_1 \odot G_2$.

Theorem 14.
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ are product fuzzy graphs, then the modular product $G_1 \odot G_2 = (\alpha_1 \odot \alpha_2, \beta_1 \odot \beta_2)$ is a fuzzy graph.

Proof:

Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ be a product fuzzy graph, then it satisfies $\beta_1(u_1u_2) \le \alpha_1(u_1) \times \alpha_1(u_2) \le \alpha_1(u_1) \wedge \alpha_1(u_2)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ be a product fuzzy graph, then it satisfies $\beta_2(v_1v_2) \le \alpha_2(v_1) \times \alpha_2(v_2) \le \alpha_2(v_1) \wedge \alpha_2(v_2)$. For the verification process, we divide it into three cases, namely

Case 1

For $u_1u_2 \in E_1, v_1v_2 \in E_2$

$(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \beta_1(u_1u_2) \times \beta_2(v_1v_2)$
$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$= \big(\alpha_1(u_1) \times \alpha_2(v_1)\big) \times \big(\alpha_1(u_2) \times \alpha_2(v_2)\big)$
$= (\alpha_1 \odot \alpha_2)(u_1,v_1) \times (\alpha_1 \odot \alpha_2)(u_2,v_2)$
$\le (\alpha_1 \odot \alpha_2)(u_1,v_1) \wedge (\alpha_1 \odot \alpha_2)(u_2,v_2)$

Case 2

For $u_1u_2 \notin E_1, v_1v_2 \notin E_2$

$(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$
$= \big(\alpha_1(u_1) \times \alpha_2(v_1)\big) \times \big(\alpha_1(u_2) \times \alpha_2(v_2)\big)$
$= (\alpha_1 \odot \alpha_2)(u_1,v_1) \times (\alpha_1 \odot \alpha_2)(u_2,v_2)$
$\le (\alpha_1 \odot \alpha_2)(u_1,v_1) \wedge (\alpha_1 \odot \alpha_2)(u_2,v_2)$

Based on case 1, and case 2 we obtain $(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) \le (\alpha_1 \odot \alpha_2)(u_1,v_1) \wedge (\alpha_1 \odot \alpha_2)(u_2,v_2)$ such that the modular product of the product fuzzy graph $G_1 \odot G_2 = (\alpha_1 \odot \alpha_2, \beta_1 \odot \beta_2)$ is a fuzzy graph.

Theorem 15.
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ are strong product fuzzy graphs, then the modular product $G_1 \odot G_2 = (\alpha_1 \odot \alpha_2, \beta_1 \odot \beta_2)$ is a strong.

Proof:

Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ be a strong product fuzzy graph, then it satisfies

$\beta_1(u_1u_2) = \alpha_1(u_1) \times \alpha_1(u_2)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ be a strong product fuzzy graph, then it satisfies $\beta_2(v_1v_2) = \alpha_2(v_1) \times \alpha_2(v_2)$. For the verification process, we divide it into three cases, namely

Case 1

For $u_1u_2 \in E_1, v_1v_2 \in E_2$

$(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \beta_1(u_1u_2) \times \beta_2(v_1v_2)$

$= \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$

$= \big(\alpha_1(u_1) \times \alpha_2(v_1)\big) \times \big(\alpha_1(u_2) \times \alpha_2(v_2)\big)$

$= (\alpha_1 \odot \alpha_2)(u_1,v_1) \times (\alpha_1 \odot \alpha_2)(u_2,v_2)$

Case 2

For $u_1u_2 \notin E_1, v_1v_2 \notin E_2$

$(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = \alpha_1(u_1) \times \alpha_1(u_2) \times \alpha_2(v_1) \times \alpha_2(v_2)$

$= \big(\alpha_1(u_1) \times \alpha_2(v_1)\big) \times \big(\alpha_1(u_2) \times \alpha_2(v_2)\big)$

$= (\alpha_1 \odot \alpha_2)(u_1,v_1) \times (\alpha_1 \odot \alpha_2)(u_2,v_2)$

Based on case 1, and case 2 we obtain $(\beta_1 \odot \beta_2)\big((u_1,v_1)(u_2,v_2)\big) = (\alpha_1 \odot \alpha_2)(u_1,v_1) \times (\alpha_1 \odot \alpha_2)(u_2,v_2)$ such that the modular product of the product fuzzy graph $G_1 \odot G_2 = (\alpha_1 \odot \alpha_2, \beta_1 \odot \beta_2)$ is a strong.

Corollary 16.
Let $G_1 = (\alpha_1, \beta_1)$ with $G_1^* = (V_1, E_1)$ and $G_2 = (\alpha_2, \beta_2)$ with $G_2^* = (V_2, E_2)$ are complete product fuzzy graphs, then the modular product $G_1 \odot G_2 = (\alpha_1 \odot \alpha_2, \beta_1 \odot \beta_2)$ is a strong.

Example 4.
Let strong product fuzzy graph $G_1 = (\alpha_1, \beta_1)$ with $\alpha_1(u_1) = 0.2$, $\alpha_1(u_2) = 0.3$, $\alpha_1(u_3) = 0.4$, $\beta_1(u_1u_2) = 0.06$, $\beta_1(u_2u_3) = 0.12$, $\beta_1(u_1u_3) = 0.08$ and product fuzzy graph $G_2 = (\alpha_2, \beta_2)$ with $\alpha_2(v_1) = 0.5$, $\alpha_2(v_2) = 0.6$, $\beta_2(v_1v_2) = 0.3$. In Figure 4, graph $G_1 \odot G_2$ is obtained as follows.

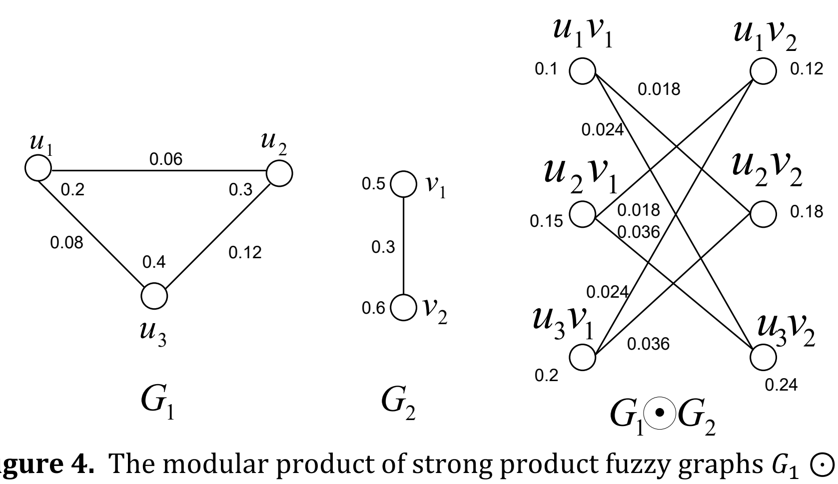

Figure 4. The modular product of strong product fuzzy graphs $G_1 \odot G_2$.

Based on Definition 13, the modular product definition of the product fuzzy graph is obtained. Furthermore, the properties of the modular product of product fuzzy graphs are established: the modular product of product fuzzy graph is a fuzzy graph as stated in Theorem 14; the modular product of strong product fuzzy graph is a strong as stated in Theorem 15; and the modular product of complete product fuzzy graph is a strong as stated in Corollary 16.

Application to Defense Communication Networks

Consider a defense communication network in which the vertices represent entities within the network. Edges represent connectivity between vertices, and edge weights represent data flow between those vertices. Suppose a defense communication network scenario requires the merging of two data flow networks. Suppose the first network, graph G_1 (central command), consists of three vertices u_1, u_2 and u_3, and the second network, graph G_2 (tactical units in the field), consists of two vertices v_1, and v_2. Suppose these two networks must be merged to obtain a new network whose vertices are formed from the first and second networks. This merger can utilize the normal product and modular product operations according to their respective characteristics.

In the normal product of product fuzzy graphs, the relationship between (u_1, v_1) and (u_2, v_2) is considered connected by an edge if it satisfies one of the following three conditions. First, if the vertices in the first graph are identical $(u_1 = u_2)$ but are adjacent in graph G_2 $(v_1v_2 \in E_2)$, second, if the vertices in the second graph are identical $(v_1 = v_2)$ but are adjacent in graph G_1 $(u_1u_2 \in E_1)$, or third, if both pairs of vertices are adjacent in their respective original graphs ($u_1u_2 \in E_1$ and $v_1v_2 \in E_2$). The advantage of the normal product of product fuzzy graphs lies in its far more comprehensive capacity to represent system interactions compared to the cartesian product or tensor product operations separately. The cartesian product of product fuzzy graphs is limited to orthogonal relationships between vertices, and the tensor product of product fuzzy graphs is limited to diagonal relationships between vertices, whereas the normal product of product fuzzy graphs combines both orthogonal and diagonal relationships simultaneously. Through the use of the normal product of product fuzzy graphs, this creates a very dense network because it combines both orthogonal and diagonal paths simultaneously namely $E(G_1 \boxtimes G_2) = E(G_1 \times G_2) \cup E(G_1 \otimes G_2)$, and in defense communication networks, this ensures maximum access path redundancy to maintain smooth data transmission.

Meanwhile, in the modular operation of product fuzzy graphs, the relationship between (u_1, v_1) and (u_2, v_2) is formed based on the principle of pattern consistency or structural consistency between the two original graphs. An edge is created if both pairs of points are adjacent to each other in their respective graphs $(u_1u_2 \in E_1, v_1v_2 \in E_2)$, or if neither point has an adjacent relationship $(u_1u_2 \notin E_1, v_1v_2 \notin E_2)$. The advantage of the modular product of product fuzzy graphs is that it serves as a tool for detecting similarities in operational patterns among various units. In other words, two units within this new network will only be connected if their relationship patterns are identical, whether they are both actively communicating or both ordered to remain inactive. This is crucial to ensure that communication channels remain synchronized between instructions from the central command and the actual movements of tactical units in the field. With this model, security resources are not distributed haphazardly but are concentrated at the most critical points that maintain the stability of the entire system. This strategy ensures the network's defense remains robust, both in securing the transmission of critical data and in maintaining confidentiality to prevent information from leaking to unauthorized units.

CONCLUSION

The normal product and modular product of product fuzzy graphs have been successfully defined. Furthermore, it has been established that the graph resulting from the normal product and modular product of product fuzzy graph is a fuzzy graph. Additionally, it has been shown that the graph resulting from the normal product and modular product of strong product fuzzy graph is strong, and similarly, the graph resulting from the normal product and modular product of complete product fuzzy graph is strong. Furthermore, in defense communication networks, the normal product of product fuzzy graph is capable of creating robust redundant paths to withstand physical disruptions, while the modular product of product fuzzy graph enhances information security by synchronizing communication patterns between units.

For future research, this could be extended to other product operations, such as homomorphic products and lexicographic products, with more comprehensive applications in defense communication networks.

AUTHOR CONTRIBUTIONS

F.F.: Conceptualization, supervision, investigation, methodology, writing draft, and formal analysis. N.R.: Conceptualization, project administration, data curation, validation, and editing. T.: Review, resources, and validation.

CONFLICT OF INTEREST

The authors declare that have no conflict of interest.

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